What are fractals?

A key characteristic of fractals is self-similarity. This means that similar structure is observed at many scales. Figure 1 illustrates the construction of a fractal known as the Koch curve. From a to d, the fractal is constructed by progressively adding copies of structure at smaller and smaller scale. A true mathematical fractal proceeds in this manner ad infinitum so that whatever magnification is used, smaller self similar structure will be observed.

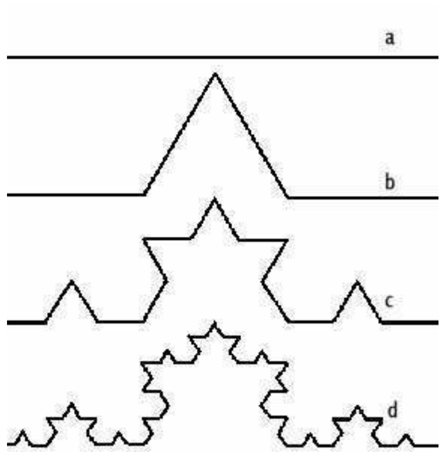


Figure 1. Construction of the Koch curve.

The Koch curve replicates exact copies of structure at every scale. This exact characteristic is referred to as geometrical self-similarity. In nature, fractals are more likely to exhibit a statistical self-similarity. The magnified parts of the fractal are statistically similar to the larger parts.

Self-similarity implies a scaling relationship. This means that the value of a property will depend on the resolution used to make the measurement. The property could be, for example, length or surface area. Referring again to Figure 1, the length of the Koch curve increases by the factor 4/3 with each iteration. Therefore, depending upon the resolution of the measurement, the length will change. Because the Koch curve is iterated ad infinitum, the distance between any two points is infinite.

Fractals in nature exhibit a similar resolution dependence. A common example is coastlines. There is no correct value for the length of a coastline. It will be determined to be greater and greater as finer and finer measuring resolution is used. As finer resolution measurements are made, the ever smaller nooks and crannies are included in the total length.

Because there is no unique measurement of a fractal property such as length, it is generally of more use to express the scaling relationship which describes the dependence of a property on resolution. Figure 2 illustrates such a relationship for the west coast of Britain as measured by Richardson and interpreted from a fractal perspective by Mandelbrot *(1,2)*. Plotting the log of coastline length as a function of the log of measurement resolution yields a straight line.

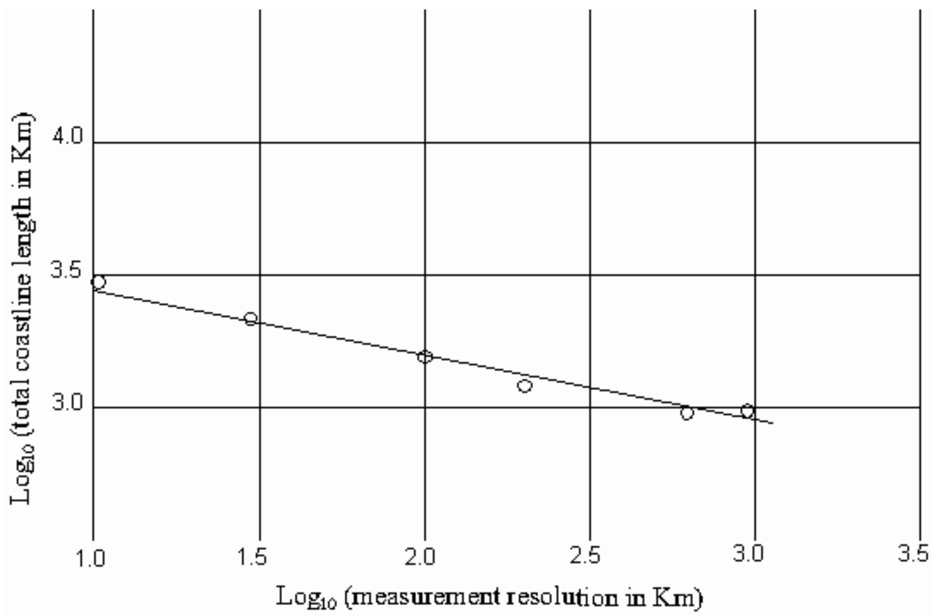


Figure 2. Length of the west coast of Britain increases with measurement resolution.

Note that the coastline length is not simply increasing linearly as measurement resolution is made finer. Rather, the length increase follows a power law. The following power function specifies the length of coastlines quite well:

(1) $L(r) = kr^{1-D}$

$L(r)$ = the total coastline length as a function of measurement resolution

r = the measurement resolution

k, D = constants

Power law scaling is characteristic of fractals. Therefore, a relationship which yields a straight line on log-log coordinates can often identify an object or phenomena as fractal.

The self-similarity and scaling characteristics of fractals can be quantitatively measured by the use of the fractal dimension. Recall that the familiar Euclidean dimension is an integer 1,2, or 3 used to denote a line, surface, or volume respectively. In comparison, the fractal dimension provides a measure of how densely an object fills space or how many new parts of an object are observed as measurement resolution becomes finer. The fractal dimension can be integer or non-integer and is always greater than the ordinary Euclidean dimension for a given object. For coastlines, the constant D in equation 1 is the corresponding fractal dimension. Note that 1-D is simply the slope of the line in Figure 3. Consequently, the west coast of Britain has a fractal dimension of ~ 1.3. This number, which is independent of the measuring units, provides quantitative scaling information. It can be interpreted as a measure of the coastline roughness or irregularity. In comparison with Britain, the coastline of Norway has a fractal dimension of ~ 1.52 *(3).* This means that the coastline of Norway has a more jagged geometry and fills a plane to a greater extent than the line describing the west coast of Britain.

There are several different fractal dimensions which can be measured and they all have somewhat different meanings. However, to further provide a sense for the concept of fractal dimension, we will discuss the simplest which is the self-similarity dimension. This dimension only applies to geometrically self-similar objects like the Koch curve. For its determination, the number N of smaller objects or pieces is counted when an object is magnified by some factor M. By magnification we mean specifically the factor needed to bring the smaller objects to exactly the same size as the object from which they were generated. The fractal dimension is then determined from:

(2) $N = (M)^D$

N = the number of new copies of an object observed after magnification.

M = the factor by which the original object must be magnified to see the new copies.

D = the fractal dimension.

This can be rewritten:

(3) $D = \log (N) / \log (M)$

Figures 3 and 4 illustrate how fractal dimension effects space filling density.

Figure 3. Object with fractal dimension = 2. Five iterations shown.

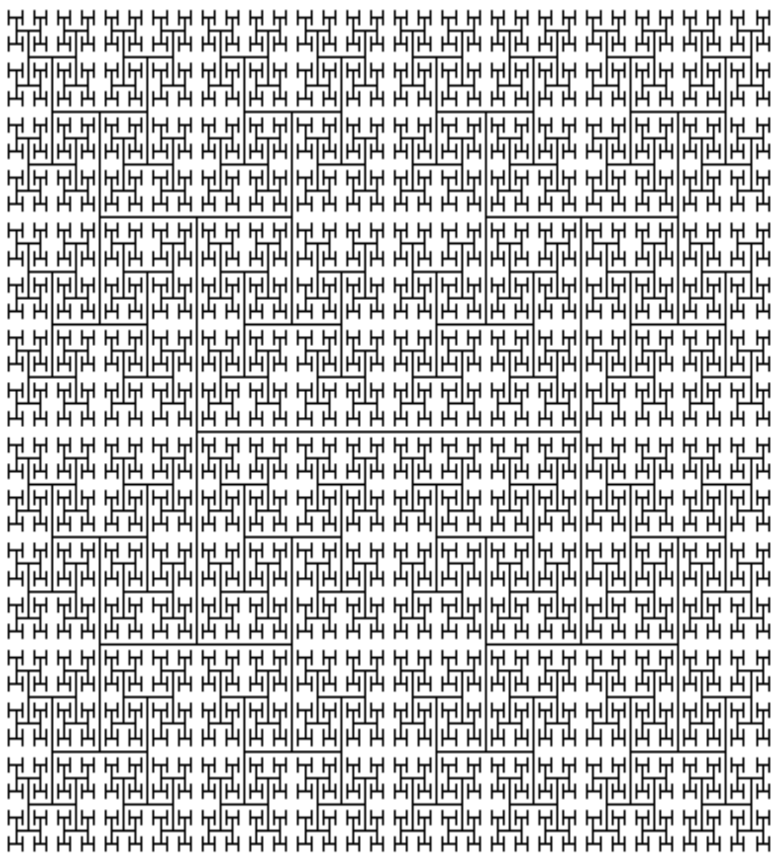
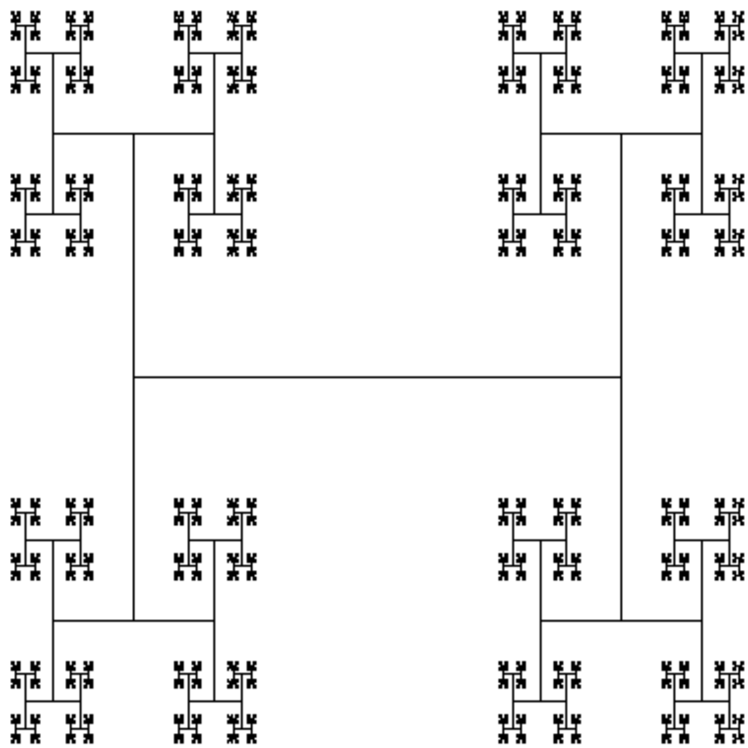


Figure 4. Object with fractal dimension = 1.26 has a low space filling density.



The starting object, called an initiator, and an "H" in this case, is iterated 5 times. Because we are considering proper mathematical fractals it is necessary to imagine the objects iterated ad infinitum. In Figure 3, four new and exact copies are produced each time an H is magnified by 2. Therefore, the fractal dimension is given by:

(4) $D = \log(4)/\log(2) = 2.0$

So Figure 3 is a line with a Euclidean dimension of 1 and a fractal dimension of 2. The line is so dense, it completely fills an area. Note that the fractal dimension provides an appropriate description of the line's space filling characteristics. For comparison, in Figure 4, a magnification of 3 is required to view the four smaller copies replicated at each iteration. This results in a lower space filling density and a correspondingly lower measure of fractal dimension:

(5) $D = \log(4)/\log(3) = 1.26$

It is important to note what the fractal dimension does not describe. It does not uniquely specify an object. For example, the Koch curve in Figure 1 and the line in Figure 4 both have the same fractal dimension (1.26).

Fractal analysis is a practical empirical tool. It provides a means for interpretation of data when fractal structure is present. Fractal analysis is used widely to classify and describe objects*(4-9)*. It can also lead to proposals concerning the mechanisms underlying physical processes.

Fractals are found throughout nature. They are present in inorganic structures such as clouds and coastlines and in living structures, such as the circulation or intestinal systems in mammals *(10)*. In living systems, this complex geometry has evolved to provide efficient solutions to a number of difficult problems, often related to fluid handling. This observation by itself suggests consideration of fractal structure to solve related engineering problems.

Finally, although we have referred to the fractal characteristics of objects, fractal analysis can equally be applied to processes in time.

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